Mathematics with Further Mathematics

Year 12 Summer Task

Welcome to Mathematics at St Peter's. The purpose of this booklet is to introduce you to the subject that you have chosen to study with us at A Level.

The Mathematics department aims to develop your logic and problem-solving skills whist introducing you to more complex mathematical material through the study of Pure Mathematics and showing the versatility of this subject through the applications of Mechanics and Statistics.

Successful study in the sixth form is all about developing the right skills, preparation, independent learning and self-motivation. We do all we can to support you in your learning and would like you to do the same. In preparation for the first term of study please complete the following:

Tasks

As you will be studying Mathematics A Level in Year 12 and then Further Mathematics A Level in Year 13, it is more important than ever to have a smooth transition into your new studies. Many of the concepts in your course will build on content you've learned at GCSE level.

We would like you to complete **5 GCSE Extension tasks** that take topics you already have knowledge of further than you're used to. We don't expect you to know everything, the task is supposed to challenge you. How to work through problems effectively when the solution isn't obvious is one of the most important skillsets that a mathematician can develop.

All work or any questions about the work can be sent to poa@st-peters.bournemouth.sch.uk at any date before September or work can be submitted on the first in school.

Equipment

In order that you are prepared for your studies next year please have the following things ready for the start of term.

1. An A4 folder for your notes, class handouts and written assignments. Also buy dividers for different topics.
2. A4 lined paper.
3. Calculator: The Casio Classwiz scientific Calculator (**CASIO 991EX**) is the standard model used for A-Level Maths and Further Maths. A Graphical Calculator like the **Casio FX-CG50** would be preferable as it includes many extra features that are especially useful for Further Maths. However, this can be expensive to buy and we might be able to buy them cheaper in bulk as a school at some later date.

Enjoy your summer – we look forward to working with you in Year 12

Graphs of quadratic equations

What you should know

How to complete the square on a quadratic expression.
How to use Venn diagrams.

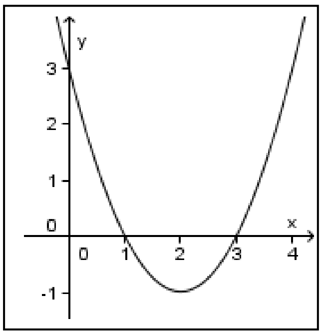
New idea

The graph of $y = (x + a)^2 + b$ has a minimum at $(-a, b)$.

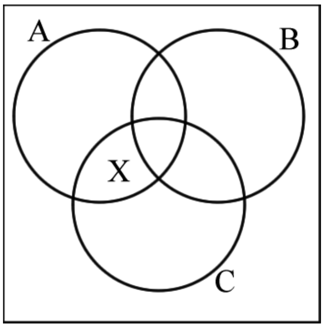
Task: Categorising quadratic curves

Think about all quadratic curves with equation $y = (x + a)^2 + b$.
Think also about these three properties.

A: The turning point has a positive x-value.
B: The turning point has a positive y-value.
C: The y-intercept is positive.



The quadratic curve shown on the right satisfies properties A and C but not B.
Its equation can be written as $y = (x - 2)^2 - 1$.
You could write this in the region marked with an X in the Venn diagram below.



- Copy the Venn diagram.
- Can you find a quadratic which doesn't satisfy any of the properties? Write this in the region outside the three circles.
- Can you find one equation for each of the other six regions?
- Is it possible to find an equation for every region?

Take it further

Find three other properties A, B and C for which all eight regions can be filled in.

Where this goes next

At A level you will learn more about the usefulness of completing the square in Core Mathematics. You will also learn another method for finding maximum and minimum points on cubics and other curves.

Interpreting graphs

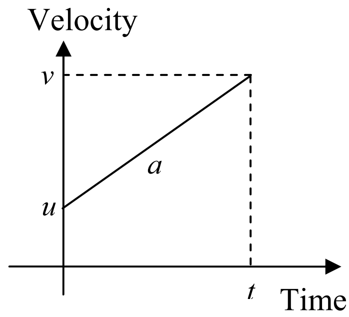
What you should know
A graph of the form $y = mx + c$ represents a straight line where m is the gradient of the line and c is the value of the y-intercept.
How to draw and interpret velocity–time graphs.

New ideas
A simple equation for an object travelling with constant acceleration is $v = u + at$, where u is its initial velocity (speed in a given direction), v is its final velocity, a is its acceleration and t is the time from the beginning to the end of its motion.
The area under a velocity–time graph represents the displacement (distance travelled in a given direction).

Task: Velocity–time graphs

A car accelerates from 10 ms^{-1} at a constant rate when leaving a built-up area.
It takes 6 seconds to reach a velocity of 22 ms^{-1} (about 50 miles per hour).

- Draw a velocity–time graph to represent this motion.
- The graph should be a straight line. You already know the equation $y = mx + c$ to describe a straight line. If you use $v = u + at$ to describe the motion, what properties would v, t, u and a represent on the graph?
- Use the equation $v = u + at$ to find the car's acceleration.
  What is the gradient of the line you have drawn?
- The area under a velocity–time graph represents displacement.
  The letter s is used for displacement.
  Find the displacement, s, of the car for this journey.
- The diagram shows the general graph for motion with constant velocity. Find a formula for the displacement, s, in terms of u, v and t.
  Check that this formula works for the example above.



Take it further

- Try substituting $v = u + at$ into your formula for s and simplifying it.
  This should give you a formula for s, in terms of u, a and t.
  Check that this formula works for the example above, too.
- A dropped object falls to Earth with an acceleration of approximately 10 ms^{-2} and has initial velocity, u, of 0.
  Use the formulae you found to work out how far it would fall and what its velocity would be after 1 second, 2 seconds, 5 seconds, 10 seconds, etc.
  What is the problem with this model?

Where this goes next
At A level constant acceleration formulae are studied in Mechanics.

Simultaneous equations

What you should know

How to solve a pair of simultaneous equations in two unknowns by substitution.

For example: $2x + 3y = 5$
$x + y = 2$

New idea

If you have three equations in three unknowns you can solve them by using substitution twice.

Task: Simultaneous equations in three unknowns

Example

Solve these simultaneous equations.

$3x + 4y + z = 3$ (1)
$x + y + z = 2$ (2)
$2x + y - z = 2$ (3)

Rewrite equation (3) as $2x + y - 2 = z$ and substitute this into the first two equations.

(1) $3x + 4y + 2x + y - 2 = 3$
which simplifies to
$5x + 5y = 5$ or $x + y = 1$ (4)

(2) $x + y + 2x + y - 2 = 2$
which simplifies to $3x + 2y = 4$ (5)

Equation (4) can be rewritten as $y = 1 - x$.
Substituting $y = 1 - x$ into equation (5) gives $3x + 2(1 - x)$ which simplifies to $x + 2 = 4$, so $x = 2$.

Substituting $x = 2$ into $y = 1 - x$ gives $y = -1$.

Substituting $x = 2$ and $y = -1$ into $2x + y - 2 = z$ gives $z = 1$.

You can check these three values by substituting them into the three original equations.

- Make up values for x, y and z. For example, $x = 3$, $y = 1$ and $z = -4$.
- Make up three expressions in x, y and z. For example, $x + y + z$, $2x + 3y - z$ and $3x - y + z$.
- It is important that your expressions are not the same as or multiples of each other, such as $x + y + z$ and $2x + 2y + 2z$.
- Work out what your expressions equal when you substitute in your values for x, y and z.
  e.g. $x + y + z = 0$
  $2x + 3y - z = 13$
  $3x - y + z = 4$
- Give these equations to another student and ask them to find your values for x, y and z.

Take it further

- In the task, it was important that your expressions were not the same or multiples of each other. Can you explain why?
- Investigate what a set of three simultaneous equations look like using a 3D graph plotter.

Where this goes next

At A level techniques for dealing with lots of simultaneous equations with many variables involve vectors and matrices which are covered in Core Mathematics and Further Mathematics.

Trigonometry 2

What you should know

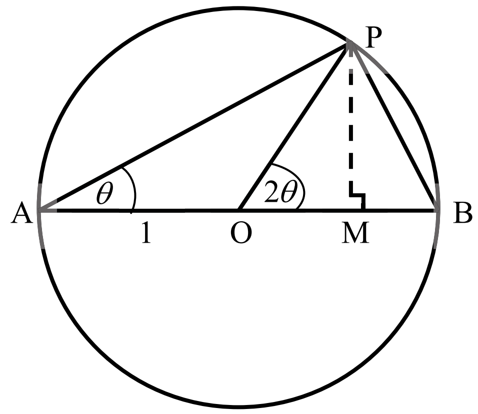
The angles in a triangle add up to 180°.
How to find sides and angles in a triangle using sine and cosine.
The circle theorems.

New idea

It is possible to use the values of sin 13° and cos 13° to work out the values of sin 26° and cos 26°. To do this you need to use the double-angle formulae. In this worksheet you will discover these formulae yourself.

Task: The double-angle formulae

The diagram shows a circle with centre O and radius 1.
AB is a diameter of the circle and P is a point on the circle.
The angle PAB is θ.



- Explain why angle APB = 90° and why angle POB = 2θ.
- Explain why AP = 2 cos θ.
- Copy the diagram. Look at triangle POM.
  Write on your diagram the lengths of the three sides in terms of the angle 2θ.
  You will need to use some trigonometry.
- Now try to find as many other edge lengths as you can. You might want to draw copies of the right-angled triangles APM, PBM and ABP. Your aim is to find expressions for the lengths PM and OM which are different from those you found earlier.
- You should have found that PM = $\sin 2\theta$ and PM = $2 \sin \theta \cos \theta$.
  Now, using your calculator, check that these are the same for a few different values of θ. Then do the same for OM. **NB:** $\sin 2\theta$ means $\sin (2\theta)$.

Take it further

- You might know these exact values: $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.
  Can you use these to find sin 60° and cos 60°?
  How about sin 15° and cos 15°?
- The diagram looks like 2θ has to be less than 90°.
  What happens for larger angles?
- Starting with the graphs of $y = \sin \theta$ and $y = \cos \theta$, sketch the graphs of $y = \sin^2 \theta$, $y = \cos^2 \theta$, $y = \sin \theta \cos \theta$, $y = \sin 2\theta$ and $y = \cos 2\theta$.
  Explain any links you notice. **NB:** $\sin^2 \theta$ means $(\sin \theta)^2$.
- Find a formula for $\tan 2\theta$ in terms of $\tan \theta$.

Where this goes next

At A level you will discover more relationships between sine, cosine and tangent in Core Mathematics.

Circles

What you should know

The equation of a circle, centre the origin and radius r is $x^2 + y^2 = r^2$.

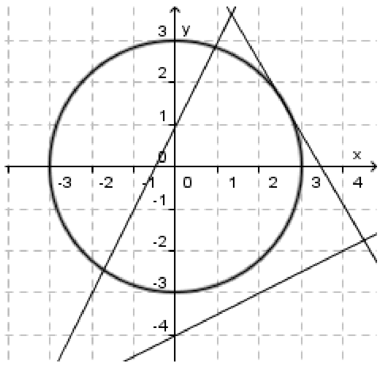
The tangent to a circle is perpendicular to the radius at the point of contact.

In a right-angled triangle, $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$.

New idea

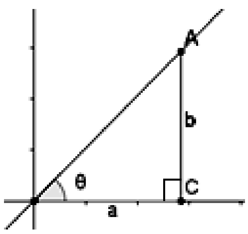
Any straight-line graph will cross a circle 0 or 1 or 2 times. If the straight line crosses the circle exactly once, it is a tangent to the circle.

A straight line with gradient m makes an angle θ with the x-axis where $m = \tan\theta$.



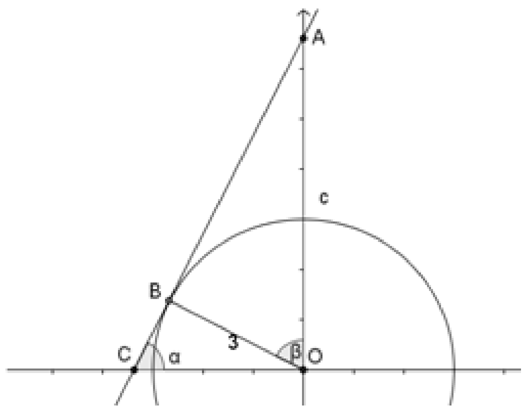
Task: Finding a tangent to a circle

- Look at the diagram.



  What is the gradient of the line joining point A to the origin? What is $\tan\theta$?
- Use a graphical calculator or a graph-drawing package on a computer to draw the circle $x^2 + y^2 = 9$ and lines with gradient 2. Can you find the equation of a line that touches the circle?
- For gradient 2, there are two possible tangents. One crosses the y-axis at c. Where does the other one cross the y-axis?

- Look at the diagram.



What is $\tan\alpha$?
Explain why angle α and angle β are equal.
How long is AB?
What is the value of c?

Take it further

Now that you know c, can you find the equations of lines that do not cross the circle?
What if the gradient of the tangent is a value different from 2?

Where this goes next

At A level the intersections of lines and curves are studied further in Core Mathematics.